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METHODS OF INVESTIGATING THE SOLVABILITY OF SYSTEMS OF LINEAR EQUATIONS OF THERMOANEMOMETRY

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Use of thermoanemometry for the investigation of turbulent flows often leads to systems of linear equations that are difficult to solve. A numerical method of solution, in which measurement errors are taken into account approximately, is proposed for the investigation of solvability of such systems of equations.

As we know, the heat emission from a heated filament to its surrounding medium depends on the modulus and direction of the vector of the velocity relative to the filament, as well as on the temperature of the filament and the medium. For sufficiently small velocities of the flow it depends on the orientation of the filament in the field of the force of gravity and to a small degree on the construction of the probe and the static pressure of the medium (see, for example, [1-4]). The value of the electric voltage at the output of the thermoanemometer characterizes the heat exchange of the filament and therefore, as a rule, is a function of several parameters corresponding to the cases mentioned above. When carrying out and processing measurements we first and foremost pursue the objective by means of different methods of measurement – for example, using multifilament transducers or successively carrying out measurements with different positions of the filament in a stationary flow – to obtain a system of equations which is solvable relative to the individual parameters of flow that are of interest to us.

The present work contains analytical investigations on the basis of the so-called cosine law [1] and investigations by the least squares method [5-8] of the matrices of systems of linear equations of thermoanemometry. Separate consideration is given to the methods of measuring with a single-filament transducer to determine the average velocity vector and Reynolds stresses in stationary turbulent flows which in the general case are three-dimensional. It is assumed that the fluid is Newtonian, isothermal, homogeneous, and incompressible. The basic propositions of these investigations are presented in [9-12] and will be repeated here to the extent which is necessary for the understanding of the present work.

1. Application of the "Cosine Law"

A quadratic approximation of the three-dimensional calibration characteristic of the probe according to [11] leads to the following relation between the single-point moments of the velocity field and the output voltage of the thermoanemometer:

$$\left(\hat{a}_i + \hat{b}_{ij}\,\frac{\Delta\hat{w}_j}{\hat{c}}\right)\frac{\Delta\hat{w}_i}{\hat{c}} + \hat{b}_{ij}\,\frac{\overline{w_i' w_j'}}{\hat{c}^2} = \Delta F \quad (i,\ j = 1,\ 2,\ 3), \tag{1}$$

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$$\left(\hat{a}_i \hat{a}_j + 4\hat{a}_i \hat{b}_{jk} \frac{\Delta\hat{w}_k}{\hat{c}} + 4\hat{b}_{ik}\hat{b}_{jl} \frac{\Delta\hat{w}_k \Delta\hat{w}_l}{\hat{c}^2} - \hat{b}_{ij}\hat{b}_{kl} \frac{\overline{w_k' w_l'}}{\hat{c}^2}\right)$$

$$\times \frac{\overline{w_i' w_j'}}{\hat{c}^2} + \left(2\hat{a}_i \hat{b}_{jk} + 4\hat{b}_{ij}\hat{b}_{kl} \frac{\Delta\hat{w}_l}{\hat{c}}\right) \frac{\overline{w_i' w_j' w_k'}}{\hat{c}^3} + \hat{b}_{ij}\hat{b}_{kl} \frac{\overline{w_i' w_j' w_k' w_l'}}{\hat{c}^4} = \overline{F'^2} \quad (i,\ j,\ k,\ l = 1,\ 2,\ 3). \tag{2}$$

Summation is carried out with respect to the repeated indices.

The linear and quadratic sensitivity coefficients $\hat{a}_i$ and $\hat{b}_{ij}$ are functions of $\hat{\alpha}_y$. This angle can be varied in succession (see Fig. 1) by rotating the probe about its axis. The corresponding sequence of measurements $\Delta\overline{F}$ and $\overline{F'^2}$ forms a system of equations with the parameter $\hat{\alpha}_y$. The solution of this nonlinear system can be reduced by an iterative method to the solution of a linear system of equations, where in the first iterations we shall neglect the nonlinear terms within parentheses in the system of equations (1), (2). The solvability of systems of linear equations obtained in this way is subsequently to be investigated.

For the case of a transducer with an inclined filament, with the condition that the heat emission of the filament depends only on the velocity component perpendicular to the filament (see [9]), we obtain

$$\frac{B(\hat{\alpha}_y;\ \hat{\beta}_y)}{\bar{B}} = \frac{B_w}{\bar{B}} [A_0(\hat{\beta}_y) + A_1(\hat{\beta}_y)\cos\alpha_y + A_2(\beta_y)\cos 2\hat{\alpha}_y]^{\frac{n}{2}}, \tag{3}$$

where

$$A_0(\hat{\beta}_y) = \frac{1}{4}\cos^2\hat{\beta}_y(3\cos 2\gamma + 1) + \sin^2\gamma;$$
$$A_1(\hat{\beta}_y) = \frac{1}{2}\sin 2\gamma \sin 2\hat{\beta}_y, \tag{4}$$
$$A_2(\hat{\beta}_y) = -\frac{1}{2}\sin^2\gamma\cos^2\hat{\beta}_y.$$

Using the relations for linear sensitivity coefficients given in [12], and omitting for the sake of simplicity the notation (^) for the point of expansion (with the exception of the example at the end of the paper), from (3) and (4) we obtain

$$a_1 = \frac{B_w}{\bar{B}} \frac{n}{2} \left[\frac{B(\alpha_y;\ \beta_y)}{B_w}\right]^{1-\frac{2}{n}} \left\{\frac{\partial A_0}{\partial\beta_y} + \frac{\partial A_1}{\partial\beta_y}\cos\alpha_y + \frac{\partial A_2}{\partial\beta_y}\cos 2\alpha_y\right\},$$
$$a_2 = \frac{B_w}{\bar{B}} \frac{n}{2} \left[\frac{B(\alpha_y;\ \beta_y)}{B_w}\right]^{1-\frac{2}{n}} \{2A_0 + 2A_1\cos\alpha_y + 2A_2\cos 2\alpha_y\}, \tag{5}$$
$$a_3 = \frac{B_w}{\bar{B}} \frac{n}{2} \left[\frac{B(\alpha_y;\ \beta_y)}{B_w}\right]^{1-\frac{2}{n}} \left\{\frac{A_1}{\cos\beta_y}\sin\alpha_y + \frac{2A_2}{\cos\beta_y}\sin 2\alpha_y\right\}.$$

The expression

$$b_{11} = \frac{n}{2} \frac{B_w}{\bar{B}} \left[\frac{B(\alpha_y;\ \beta_y)}{B_w}\right]^{1-\frac{4}{n}} \sum_{k=0}^{4} B_k^{(11)}(\beta_y)\cos(k\alpha_y) \tag{6}$$

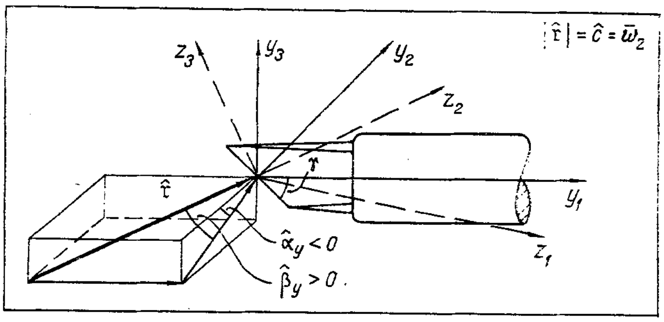

Fig. 1. Diagram of a single-filament transducer. The heated filament lies in the y_1y_2 plane; the z_3 axis lies in the y_2y_3 plane. $|\hat{\tau}| = \hat{c} = \bar{w}_2$.

is given only to explain the structure of the quadratic sensitivity coefficients. The coefficients entering into this expression have the form

$$B_0^{(11)}(\beta_y) = A_0^2 + \frac{1}{2} A_1^2 + \frac{1}{2} A_2^2 + P\left[\left(\frac{\partial A_0}{\partial \beta_y}\right)^2 + \frac{1}{2}\left(\frac{\partial A_1}{\partial \beta_y}\right)^2 + \frac{1}{2}\left(\frac{\partial A_2}{\partial \beta_y}\right)^2\right] + \frac{1}{2} A_0 \frac{\partial^2 A_0}{\partial \beta_y^2} + \frac{1}{4} A_1 \frac{\partial^2 A_1}{\partial \beta_y^2} + \frac{1}{4} A_2 \frac{\partial^2 A_2}{\partial \beta_y^2} \qquad (7)$$

and they depend not only on the functions $A_0(\beta_y)$, $A_1(\beta_y)$, and $A_2(\beta_y)$, and their first and second derivatives with respect to β_y, but also on the modulus of the velocity c:

$$P = \frac{1}{2}\left(\frac{n}{2} - 1\right) - \frac{n}{8} \frac{\left(\frac{c^n \tilde{B}}{E_0^2}\right) \frac{B(\alpha_y;\ \beta_y)}{\tilde{B}}}{1 + \left(\frac{c^n \tilde{B}}{E_0^2}\right) \frac{B(\alpha_y;\ \beta_y)}{\tilde{B}}}. \qquad (8)$$

Neglecting the terms with quadratic sensitivity coefficients in Eqs. (1) and (2), we first consider the case of linear approximation of the calibration characteristic of the transducer. Symmetry of the coefficients a_i relative to α_y allows us to separate the systems obtained from (1) and (2) (see [10, 11]), which leads to the necessity of investigating the properties of the following four matrices:

$$\mathfrak{G}_{GI} = \{\vec{a}_1,\ \vec{a}_2\}, \qquad (9a)$$

$$\mathfrak{G}_{GII} = \{\vec{a}_3\}, \qquad (9b)$$

$$\mathfrak{G}_{RI} = \{\vec{a}_1^2,\ \vec{a}_2^2,\ \vec{a}_3^2,\ \overrightarrow{2a_1a_2}\}, \qquad (9c)$$

$$\mathfrak{G}_{RII} = \{\overrightarrow{2a_1a_3},\ \overrightarrow{2a_2a_3}\}. \qquad (9d)$$

It is obvious that the expressions in front of the curly brackets in Eq. (5) have no effect on the relation between the column vectors of the matrices (9). The column vectors of interest to us are in essence a linear combination of "trigonometric" vectors. In the case of the matrix (9c) they have, for example, the form $\overrightarrow{\cos(m\alpha_y)}$ for m = 0, 1, 2, 3, 4. It can be shown that these five trigonometric vectors (we shall call them base vectors) are linearly independent, with certain simple exceptions. For example, for a transducer with inclined filament we find that for $0 < \gamma < \pi/2$ and $\gamma - \pi/2 < \beta_y < \pi/2$ ($\beta_y \neq 0$) the number of base vectors serving for the formation of column vectors is greater than the number of columns. Hence, it follows that linear dependence exists only under certain conditions which must be satisfied by the coefficients of the base vectors. Subsequently, we consider cases in which the number of base vectors in the matrices under consideration is less than the number of column vectors, i.e., cases where linear dependence exists. Thus, for the case of a normal transducer for $\gamma = \pi/2$ and $-\pi/2 \leq \beta_y \leq \pi/2$ ($\beta_y \neq 0$) the column vectors of the matrix (9c) are made up only of the three base vectors $\vec{1}$, $\overrightarrow{\cos 2\alpha_y}$, $\overrightarrow{\cos 4\alpha_y}$. Linear dependence must exist for four columns made up of these three base vectors. Similar arguments can be developed for the remaining three matrices (9) and for the other angles γ and β_y.

The matrices obtainable in the case of quadratic approximation of the calibration characteristic of the transducers from Eqs. (1) and (2), when we neglect the nonlinearities and use the symmetry condition of the sensitivity coefficients, will not be investigated here in detail (see [11]).

Important is the fact that under the assumption

$$Q = n - 1 - n \frac{c^n \tilde{B}}{2E^2} \frac{B(\alpha_y;\ \beta_y)}{\tilde{B}} \approx \text{const} \qquad (10)$$

the corresponding linear and quadratic sensitivity coefficients become approximately proportional to one another: $\vec{a}_2 \sim \vec{b}_{22}$, $\vec{a}_1 \sim \vec{b}_{12}$, $\vec{a}_3 \sim \vec{b}_{23}$. For the commonly adopted conditions of the investigations

$$\left(\frac{\tilde{B}}{B_w} = 1;\quad 0.5 \lesssim \frac{B(\alpha_y;\ \beta_y)}{\tilde{B}} \leqslant 1;\quad \frac{E_0^2}{c^n \tilde{B}} \approx 0.8;\quad n = 0.5;\quad c \approx 12 \cdot \frac{\text{m}}{\text{sec}}\right)$$

we obtain

$$-0.618-0.021 \lesssim Q \lesssim -0.618+0.021,$$

which means that the greatest relative deviation of this function from its mean value reaches 3.5%.

The presence of these approximate proportionalities between the linear and quadratic sensitivity coefficients is confirmed by calculations, and this leads to unfavorable conditioning properties of the matrices obtained from (1).

Analogously to Q, for the expression P from Eq. (8) we have

$$-0.405-0.005 \lesssim P \lesssim -0.405+0.005,$$

i.e., it is approximately a constant quantity. In the case of quadratic approximation of the calibration characteristics of the transducers, the column vectors of the matrices on the basis of (6) and (7) can be reduced to a linear combination of base vectors. On the whole, the method of comparing the number of base vectors with the number of columns points to rigid constraints on the solvability conditions of the systems of equations obtained from (2).

2. Application of the Least Squares Method

In the first part it was shown that in the solution of the (overdetermined) systems of linear equations which result when the method of measurement presented above is used, in certain cases difficulties arise, since the column vectors of the coefficient matrices can be linearly dependent. Thoughtless application of the least squares method can then lead to physically meaningless results. Therefore, below we present a simple numerical method of checking linear independence of column vectors; in other words, we establish the solvability of systems of linear equations. The objective of applying the numerical method consists of the following: in the case of linear dependence of the column vectors $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n$ we find one of the bases $\{\vec{s}_{i_1}, \vec{s}_{i_2}, \ldots, \vec{s}_{i_k}\}$ ($k < n$; $1 \le i_1, i_2, \ldots, i_k \le n$) of the linear space R' ($R' \subseteq R^m$) formed by the column vectors $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n$ [13]. For each vector $\vec{s}_i$ ($1 \le i \le n$; $i \ne i_1, i_2, \ldots, i_k$) we calculate its coordinates $c_i^{(\nu)}$ ($1 \le \nu \le k$) to this base:

$$\vec{s}_i = \sum_{\nu=1}^{k} c_i^{(\nu)} \vec{s}_{i_\nu}. \tag{11}$$

We next determine whether the system of equations

$$\sum_{i=1}^{n} \vec{s}_i x_i = \vec{s}_{n+1} \tag{12}$$

is solvable and in the affirmative case find all solutions. The solutions of the system (12) found by the least squares method below will be called "generalized solutions." For the application of this method to thermoanemometry it is particularly important to take into account, although approximately, the influence of random errors of measurement of the vectors $\vec{s}_i$ ($1 \le i \le n+1$). We show how this can be done. To begin we assume that s_{ij} are found exactly. Then the numerical method consists of successive solution of problems in a form that is close to the one under consideration, for example, in [13]:

1) to begin $M = \emptyset$, $h = 1$. We choose the index i_1 ($1 \le i_1 \le n$);

2) $\vec{s}_{i_h}$ is added to M;

3) for each i ($1 \le i \le n+1$; $i \ne i_1, i_2 \ldots i_h$) we determine the coefficients $c_i^{(\nu)}$ ($1 \le \nu \le h$) in such a way that for the vector

$$\vec{s}_i' = \sum_{\nu=1}^{h} c_i^{(\nu)} \vec{s}_{i_\nu} \tag{13}$$

the quantity

$$Q_i^{(h)} = (\vec{s}_i - \vec{s}_i')^2 = \sum_{j=1}^{m} \left[s_{ij} - \sum_{\nu=1}^{h} c_i^{(\nu)} s_{i_\nu j} \right]^2 \tag{14}$$

is a minimum;

4) only two cases are possible:

4a) $h = n$: $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n$ are linearly independent, the computation process is terminated;

4b) $h < n$, two cases exist:

4ba) for all i, when $1 \leq i \leq n$, $i \neq i_1, i_2, \ldots, i_k$,

$$Q_i^{(h)} = 0 \tag{15}$$

holds, and R' has the dimension $k = h$. Vectors from M form one of the bases of the space R'. The coefficients for the expression (11) have been found. The computation process is terminated;

4bb) at least one i ($1 \leq i \leq n$; $i \neq i_1, i_2, \ldots, i_h$) exists for which

$$Q_i^{(h)} > 0; \tag{16}$$

5) i_{k+1} is chosen from natural numbers p for which

$$Q_p^{(h)} = \max_{\substack{1 \leqslant i \leqslant n \\ i \neq i_1, \ldots i_h}} Q_i^{(h)} \tag{17}$$

is valid. Then h is increased by one. The computation process is continued from the point 2). It can easily be shown that k linearly independent vectors $\vec{s}_{i_1}, \vec{s}_{i_2}, \ldots, \vec{s}_{i_k}$ will be found if and only if R' has the dimensionality k.

3. Application to a System of Linear Equations

Let $\{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_k\}$ ($k \leq n$) be one of the bases of the space R'. We denote

$$z_j = x_j + \sum_{\nu=k+1}^{n} c_\nu^{(j)} x_\nu \quad (1 \leqslant j \leqslant k); \tag{18}$$

then (12) assumes the form

$$\sum_{\nu=1}^{k} \vec{s}_\nu z_\nu = \vec{s}_{n+1}, \tag{19}$$

where $x_1, x_2, \ldots, x_n$ is the solution of the system (12) if and only if $z_1, z_2, \ldots, z_k$ is the solution of the system (19). Since the vectors $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_k$ are linearly independent, the system (19) has a unique solution if and only if $Q_{n+1}^{(k)} = 0$. If $Q_{n+1}^{(k)} = 0$, then $z_j = c_{n+1}^{(j)}$ ($1 \leq j \leq k$) are this solution, and from

$$c_{n+1}^{(j)} = x_j + \sum_{\nu=k+1}^{n} c_\nu^{(j)} x_\nu \quad (1 \leqslant j \leqslant k) \tag{20}$$

we can obtain all solutions $x_1, x_2, \ldots, x_n$ of the system (12). The quantities $c_{n+1}^{(j)}$ ($1 \leq j \leq k$) thus found in this case are the solution of the system (12) (for $k = n$) or they are the values of linear combinations of the unknowns (for $k < n$). If $Q_{n+1}^{(k)} > 0$, then the system (19) has a "generalized" solution but no solution in the usual sense. In this case (12) also in the usual sense is unsolvable. If we consider (12) as a problem of linear regression, then $c_{n+1}^{(j)}$ in the case $m > k$ can serve as estimates for $x_1, x_2, \ldots, x_n$ or as estimates of their linear combinations (20) [6].

The knowledge of linear combinations is useful in many practical cases. For example, if we know the definite limits between which $x_{k+1}, x_{k+2}, \ldots, x_n$ must be located, then from the expression (20) we can obtain the limits for $x_1, x_2, \ldots, x_k$ (see [11]).

In the application to thermoanemometry, s_{ij} ($1 \leq i \leq n$, $1 \leq j \leq m$) denote the sensitivity coefficients [see (1), (2), (5), (6)]. They are obtained as the right side of $s_{n+1,j}$ [$\Delta\hat{F}$, F'^2, see (1), (2)] by means of a series of successive measurements. The method is to be generalized with the aim of taking into account the random errors of measurement, when answering the following question: does the expression (11) exist for $\vec{s}_i$ and is the system (12) solvable or not solvable in the usual sense? For this we use certain rough approximations which give a tentative estimate, which, however, is convenient to apply, for the solvability of the system (12) with coefficients found with a certain accuracy. Below two somewhat differing variants are presented.

1. s_{ij} ($1 \leq i \leq n+1$; $1 \leq j \leq m$), strictly speaking, are realizations of the random quantities σ_{ij}, while $\vec{s}_i$ are realizations of the random vectors $\vec{\sigma}_i$. From the analysis of the method of measurement in a majority of

cases we can obtain an estimate $\delta(s_{ij})$ for the accuracy of determining the quantity s_{ij} (see [14]). The first, naturally rough assumption is provided by $E(\sigma_{ij}) = s_{ij}$ and $D(\sigma_{ij}) = [\delta(s_{ij})]^2$. We put

$$D(\vec{\sigma}_i) = \frac{1}{m}\sum_{j=1}^{m} D(\sigma_{ij}) = \frac{1}{m}\sum_{j=1}^{m}[\delta(s_{ij})]^2. \tag{21}$$

The second assumption consists of the fact that σ_{qj} and σ_{pj} $(1 \le j \le m)$ for $p \ne q$ must be independent random quantities. For a linear combination (13) with nonrandom quantities $c_i^{(\nu)}$, according to (21), from the relation for determining the error [5], [8] we can obtain the expression

$$D(\vec{\sigma}'_i) = \sum_{\nu=1}^{h}(c_i^{(\nu)})^2 D(\vec{\sigma}_{i_\nu}) = \frac{1}{m}\sum_{\nu=1}^{h}(c_i^{(\nu)})^2\sum_{j=1}^{m}[\delta(s_{i_\nu j})]^2. \tag{22}$$

If we determine $c_i^{(\nu)}$ in such a way that $Q_i^{(h)}$ [see (14)] is a minimum, then according to (21), (22), and (14) for each i $(1 \le i \le n+1;\ i \ne i_1, i_2, \ldots, i_h)$ we can compute the value

$$u_i^{(h)} = \min\left\{\sqrt{\frac{1}{m}Q_i^{(h)}} - \sqrt{D(\vec{\sigma}'_i)},\ \sqrt{\frac{1}{m}Q_i^{(h)}} - \sqrt{D(\vec{\sigma}_i)}\right\}. \tag{23}$$

We arrive at the definition: a vector $\vec{s}_i$ $(1 \le i \le n+1;\ i \ne i_1, i_2, \ldots, i_h)$ within the limits of measurement accuracy can be represented by the linear combination (13) of the vectors $\vec{s}_{i_1}, \vec{s}_{i_2}, \ldots, \vec{s}_{i_h}$, if $u_i^{(k)} \le 0$.

2. The second variant was considered in [14]. Here instead of the quantity $u_i^{(h)}$, the value

$$v_i^{(h)} = \sqrt{\frac{1}{m-h}Q_i^{(h)}} - \sqrt{D(\vec{\sigma}_i) + D(\vec{\sigma}'_i)} \tag{24}$$

is computed, and then, just as $u_i^{(h)}$ in the first variant, it is used for the corresponding definition.

An application of these variants requires the following changes in the method presented above: (15) must be replaced by $u_i^{(h)} \le 0$ or $v_i^{(h)} \le 0$; in (16) and (17), $Q_i^{(h)}$ is substituted by $u_i^{(h)}$ or $v_i^{(h)}$.

A program in ALGOL was set up for the computations. The results of the calculations showed an insignificant difference between the two variants.

The system of equations (12) is considered "solvable within the limits of measurement accuracy" in the case of $u_{n+1}^{(k)} \le 0$ or $v_{n+1}^{(k)} \le 0$, where k, determinable by the modified method, is the "dimensionality" of the space R'. The coefficients $c_{n+1}^{(1)}, c_{n+1}^{(2)}, \ldots, c_{n+1}^{(k)}$ are considered by us for the given measurement accuracy as possible values for the unknowns $x_1, x_2, \ldots, x_n$ or their linear combinations (20). If the system of equations (12) is "not solvable within the limits of measurement accuracy," then it is assumed that this is due to the shortcomings of the theoretical model serving as our basis (for example, the linear approximation of the calibration characteristic). Then we can attempt to obtain the solution by means of the approved model (for example, a quadratic approximation of the calibration characteristic).

Example. The matrix (9c), when measuring with a normal transducer ($\gamma = \pi/2$; $\hat{\beta}_y = 17.9[°]$, $\hat{\alpha}_y = 0, 15, 30, 45, 60\,[°]$), has the form

$$\begin{aligned}
0.006\,x_1 + 0.293\,x_2 - 0.085\,x_4 &= 0.00820,\\
0.004\,x_1 + 0.283\,x_2 + 0.017\,x_3 - 0.070\,x_4 &= 0.01023,\\
0.002\,x_1 + 0.258\,x_2 + 0.064\,x_3 - 0.039\,x_4 &= 0.01618,\\
0.219\,x_2 + 0.140\,x_3 + 0.005\,x_4 &= 0.02494,\\
0.011\,x_1 + 0.171\,x_2 + 0.246\,x_3 + 0.086\,x_4 &= 0.03188.
\end{aligned} \tag{25}$$

Here $\delta(s_{ij}) = 0.012$ for $1 \le i \le 4$, $1 \le j \le 5$; $\delta(s_{4j}) = 0.00032$ $(1 \le j \le 5)$. For $i_1 = 2$ with this method we obtain $(h = 1)$ $\vec{s}'_1 = 0.017\,\vec{s}_2$, $v_1^{(1)} = -0.007$; $\vec{s}'_4 = -0.127\,\vec{s}_2$, $v_4^{(1)} = 0.052$, $\vec{s}'_3 = 0.303\,\vec{s}_2$, $v_3^{(1)} = 0.106$; $\vec{s}'_5 = +0.066\,\vec{s}_2$, $v_5^{(1)} = 0.013$.

In accordance with (17) we choose $i_2 = 3$ $(h = 2)$:

$$\begin{aligned}
\vec{s}'_1 &= 0.010\,\vec{s}_2 + 0.022\,\vec{s}_3, \quad v_1^{(2)} = -0.008,\\
\vec{s}'_4 &= -0.289\,\vec{s}_2 + 0.536\,\vec{s}_3, \quad v_4^{(2)} = -0.010,
\end{aligned} \tag{26}$$

$$\vec{s}_5' = 0.031\ \vec{s}_2 + 0.113\ \vec{s}_3,\ v_5^{(2)} = 0.0003.$$

The computation process is terminated. The result is k = 2; $\vec{s}_1$ and $\vec{s}_4$ within the limits of accuracy of measurements are represented by the linear combinations (26) (since $v_1^{(2)} \le 0$, $v_4^{(2)} \le 0$); the system of equations (25) is "not solvable within the limits of measurement accuracy" (since $v_5^{(2)} > 0$). If $v_5^{(2)} \le 0$, then in conformity with the definitions given above, each quadruple x_1, x_2, x_3, x_4, elements of which possess the properties

$$x_2 + 0.010\ x_1 - 0.289\ x_4 = 0.031,\ x_3 + 0.022\ x_1 + 0.536\ x_4 = 0.113,$$

could be called "within the limits of measurement accuracy" a possible solution of the system (25).

NOTATION

y_i, Cartesian coordinates referred to the transducer (i = 1, 2, 3) (see Fig. 1); z_i, Cartesian coordinates referred to the flow (i = 1, 2, 3) (see Fig. 1); w_i, velocity components in the system z_i (i = 1, 2, 3); $\mathfrak{v}$, velocity vector; c, modulus of velocity vector; α_y, β_y, angles of attack (see Fig. 1); γ, angle of inclination of the filament (see Fig. 1); F, dimensionless output voltage of the anemometer; $B(\alpha_y, \beta_y)$, calibration function (see [12]); $\tilde{B}$, relative value of the function $B(\alpha_y, \beta_y)$ (see [12]); n, index in the King law (see [12]); E_2^0, calibration constant in the King law (see [12]); B_W, calibration constant; a_i, linear sensitivity coefficients (i = 1, 2, 3); b_{ij}, quadratic sensitivity coefficients (i, j = 1, 2, 3); $(\bar{\ })$, time-averaged value; $(\ )'$, pulsation component; $(\hat{\ })$, notation of quantities at the point of expansion of the approximation; $\Delta(\hat{\ }) = (\bar{\ }) - (\hat{\ })$; $(\vec{\ })$, column vectors; x_i ($1 \le i \le n$), unknowns (see [12]); $\vec{s}_i$ ($1 \le i \le n$), column vectors of coefficient matrices (see [12]); $\vec{s}_{n+1}$, vector of the right side (see [12]); $\vec{s}_{ij}$ ($1 \le i \le n+1$, $1 \le j \le m$), j-th components of the vector $\vec{s}_i$; M, set of vectors; $\emptyset$, empty set; $E(\sigma)$, mathematical expectation of a random quantity; $D(\sigma)$, variance of a random quantity.

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